

Graded Lie algebras of Cartan type in characteristic 2

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Abstract

We investigate the graded Lie algebras of Cartan type W, S and H in characteristic 2 and determine their simple constituents and some exceptional isomorphisms between them. We also consider the graded Lie algebras of Cartan type K in characteristic 2 and conjecture that their simple constituents are isomorphic to Lie algebras of type H .

1 Introduction

The simple Lie algebras over the complex numbers were first classified by Killing (1888) and Cartan (1894). They fall in four infinite families A_n, B_n, C_n and D_n and five exceptional cases E_6, E_7, E_8, G_2 and F_4 . All of these Lie algebras have analogues in the modular case and thus can be defined over arbitrary fields of characteristic p . The resulting Lie algebras are called *classical* Lie algebras and they are simple for $p \geq 5$. We refer to [8, Chapter 4] for background.

Over fields of characteristic p there are also non-classical simple Lie algebras known. There are the four infinite families of *graded Cartan type*: the Witt algebras W , the special algebras S , the Hamiltonian algebras H and the contact algebras K . The Lie algebras of (*general*) *Cartan type* are deformations of these. Additionally, for $p = 5$ there exists the family of Melikian type Lie algebras. The classification of the simple Lie algebras over algebraically closed fields of characteristic $p \geq 5$ has been completed around the beginning of this century by Strade and Premet. It asserts that every simple Lie algebra over an algebraically closed field of characteristic p is either classical or of Cartan or Melikian type. We refer to [7] and [8] for background.

Over fields of characteristic p with $p \in \{2, 3\}$ the classification of simple Lie algebras is wide open. In particular for $p = 2$ it seems that many new phenomena arise and the classification will differ significantly from those in characteristic 0 and $p \geq 5$. The classical Lie algebras and the Lie algebras of Cartan type have analogues in characteristic 2, but these are not necessarily simple. The constituents of the classical Lie algebras in characteristic 2 have been determined independently by Hogeweij [3] and Hiss [2]. On the

other hand various new infinite series of simple Lie algebras in characteristic 2 have been discovered, see for example [4], [11], [6], [5] or [1].

In the main part of this note we consider the graded Lie algebras of Cartan types W, S and H in characteristic 2. We determine their simple constituents and exhibit some exceptional isomorphisms between them. As a result, we obtain that the following list of simple Lie algebras in characteristic 2 contains the simple constituents of all graded Lie algebras of Cartan types W, S and H up to isomorphism.

algebra	parameters	dimension	notes
$W(n, m)$	$n > 1$	$n2^{m_1+\dots+m_n}$	Theorem 2.4 in [9, Chap 4]
$W(1, (l))'$	$l > 1$	2^{l-1}	Theorem 2 below
$S(n, m)$	$n > 2$	$(n-1)(2^{m_1+\dots+m_n} - 1)$	Theorem 3.5 in [9, Chap 4]
$S(2, m)'$	$1 \notin m$	$2^{m_1+m_2} - 2$	Theorem 3 below
$H(n, m)$	$n > 3$ even	$2^{m_1+\dots+m_n} - 2$	Theorem 4 below

Figure 1: Simple Lie algebras

There are further isomorphisms among the Witt algebras, the special algebras and the Hamiltonian algebras known. For example, two Witt algebras $W(n, m)$ and $W(n, m')$ are isomorphic if m is a permutation of m' . Moreover, in characteristic 2 there are further isomorphisms possible. For example, computations based on the computer algebra system GAP [10] show that $H(4, (1, 1, 1, 1)) \cong S(3, (1, 1, 1))$ and $H(4, (2, 1, 1, 1)) \cong S(3, (2, 1, 1))$. This induces the following conjecture.

1 Conjecture: $H(4, (m_1, \dots, m_3, 1)) \cong S(3, (m_1, \dots, m_3))$ in characteristic 2.

In the final section of this note, we consider the graded Lie algebras of contact type K . Based on experimental evidence, we conjecture that their simple constituents are isomorphic to quotients of Lie algebras of type H .

2 Preliminaries

We briefly recall the definition of the polynomial algebras $A(n)$ and $A(n, m)$ for $n \in \mathbb{N}$ and $m \in \mathbb{N}^n$. Let \mathbb{F} be a field of characteristic $p > 0$ and let X_1, \dots, X_n be n pairwise commuting indeterminates over \mathbb{F} . For $a \in \mathbb{N}^n$ we write X^a for $X_1^{a_1} \cdots X_n^{a_n}$. Let $A(n)$ denote the commutative algebra consisting of all formal sums over \mathbb{F} of the form $\sum_{a \in \mathbb{N}^n} \alpha_a X^a$ equipped with the usual addition and the multiplication

$$\left(\sum_a \alpha_a X^a\right) \left(\sum_b \beta_b X^b\right) = \sum_c \left(\sum_{a+b=c} \alpha_a \beta_b \binom{c}{a}\right) X^c,$$

where the multi-binomial coefficient is evaluated modulo p and thus is considered as an element in the prime field of \mathbb{F} . For $m \in \mathbb{N}^n$ let $\tau = (p^{m_1} - 1, \dots, p^{m_n} - 1)$. Then

$A(n, m) = \langle X^a \mid 0 \leq a \leq \tau \rangle \leq A(n)$ is a subalgebra of dimension $p^{m_1 + \dots + m_n}$. Let D_j denote the partial derivation in X_j on $A(n, m)$.

3 The Witt algebras

The Witt algebra $W(n, m)$ is defined as the set of elements $\{\sum_{j=1}^n f_j D_j \mid f_j \in A(n, m)\}$ equipped with the usual addition and the Lie bracket

$$[f D_i, g D_j] = f D_i(g) D_j - g D_j(f) D_i + f g [D_i, D_j].$$

The set $\{X^{(a)} D_i \mid 1 \leq i \leq n, 0 \leq a \leq \tau\}$ is a basis for $W(n, m)$ and $W(n, m)$ has dimension $n \dim(A(n, m))$.

The algebra $W(1, (1))$ has dimension 2 and thus is solvable. In Theorem 2.4 of Chapter 4 in [9] is shown that $W(n, m)$ is simple if $n > 1$. We consider the remaining cases in the following theorem.

2 Theorem: *Let $\text{char}(\mathbb{F}) = 2$ and $W = W(1, m)$ with $m \neq (1)$. Then $\dim(W') = \dim(W) - 1$ and W' is simple.*

Proof: Let $m = (l)$ with $l \in \mathbb{N}, l \neq 1$. Let $a_i := X^{(i)} D_1$ for $0 \leq i \leq 2^l - 1$. Then $\{a_i \mid 0 \leq i \leq 2^l - 1\}$ is a basis for W . Let c_{ijk} denote the corresponding structure constants. Then $c_{ijk} = 0$ if $k \neq i + j - 1$ and otherwise

$$c_{ijk} = \binom{i+j-1}{i} - \binom{i+j-1}{j}.$$

Hence $[a_0, a_i] = a_{i-1}$ for $0 \leq i \leq 2^l - 1$ (with $a_{-1} := 0$). Thus $a_i \in W'$ for $0 \leq i \leq 2^l - 2$ and the structure constants also imply that $a_{2^l-1} \notin W'$. Thus W' has codimension 1 in W .

Suppose that I is a non-zero ideal in W' . Let $x = \sum_{i=0}^{2^l-1} \lambda_i a_i$ be a non-zero element in I . Let k be maximal with $\lambda_k \neq 0$. Then

$$(\text{ad} a_0)^k(x) = \sum_{i=0}^k \lambda_i a_{i-k} = \lambda_k a_0.$$

Hence $a_0 = \lambda_k^{-1} (\text{ad} a_0)^k(x) \in I$. As $[a_0, a_i] = a_{i-1}$ for $0 \leq i \leq 2^l - 1$, it follows that $W' = \langle a_0, \dots, a_{2^l-2} \rangle \leq I$ and thus $W' = I$. •

4 The special algebras

Let $n > 1$ and $m \in \mathbb{N}^n$. We define

$$\text{div} : W(n, m) \rightarrow A(n, m) : \sum_{j=1}^n f_j D_j \mapsto \sum_{j=1}^n D_j(f_j).$$

Then the special Lie algebra $S(n, m)$ is the derived subalgebra of the kernel of div . Define $D_{ij} : A(n, m) \rightarrow W(n, m) : f \mapsto D_j(f)D_i - D_i(f)D_j$. Then $S(n, m)$ is generated by $\{D_{ij}(f) \mid f \in A(n, m), 0 \leq i < j \leq n\}$ and has the dimension $(n-1)(\dim(A(n, m)) - 1)$. The algebra $S(2, (1, 1))$ is solvable. The algebras $S(n, m)$ for $n > 2$ are simple as shown in [9], Theorem 3.5 in Chapter 4. The remaining cases are considered in the following theorem. Note that $S(2, (m_1, m_2)) \cong S(2, (m_2, m_1))$ for every $m_1, m_2 \in \mathbb{N}$.

3 Theorem: Let $\text{char}(\mathbb{F}) = 2$ and $S = S(2, m)$ with $m \neq (1, 1)$.

- (a) If $1 \notin m$, then $\dim(S') = \dim(S) - 1$ and S' is simple.
- (b) If $m = (1, m_2)$, then $S'/N(S) \cong W(1, (m_2))'$.

Proof: We first investigate $S = S(2, m)$ before we consider (a) and (b). Let $x_j = X^{(0,j)}D_1$ and $y_i = X^{(i,0)}D_2$ and $z_{ij} = X^{(i+1,j)}D_1 - X^{(i,j+1)}D_2$. Then $\{x_j, y_i, z_{ij} \mid 0 \leq i \leq 2^{m_1}-2, 0 \leq j \leq 2^{m_2}-2\}$ is a basis for S . Define $x_{-1}, y_{-1} = 0$ and

$$\gamma_{ijkl} = \binom{i+k+1}{i+1} \left(\binom{j+l}{j} + \binom{j+l}{j-1} \right) - \binom{i+k+1}{i} \left(\binom{j+l}{j+1} + \binom{j+l}{j} \right).$$

Then

- $[x_i, x_j] = 0$ for $0 \leq i, j \leq 2^{m_2}-2$;
- $[y_i, y_j] = 0$ for $0 \leq i, j \leq 2^{m_1}-2$;
- $[z_{ij}, z_{kl}] = \gamma_{ijkl} z_{i+k, j+l}$ for $0 \leq i, k \leq 2^{m_1}-2$ and $0 \leq j, l \leq 2^{m_2}-2$;
- $[x_0, y_i] = y_{i-1}$ for $0 \leq i \leq 2^{m_1}-2$;
- $[y_0, x_j] = x_{j-1}$ for $0 \leq j \leq 2^{m_2}-2$;
- $[x_i, y_j] = z_{j-1, i-1}$ for $0 < i \leq 2^{m_2}-2$ and $0 < j \leq 2^{m_1}-2$;
- $[x_i, z_{0k}] = \binom{i+k+1}{i} x_{i+k}$ for $0 \leq i, k \leq 2^{m_2}-2$;
- $[x_i, z_{jk}] = \binom{i+k+1}{i} z_{j-1, i+k}$ for $0 \leq i, k \leq 2^{m_2}-2, 0 < j \leq 2^{m_1}-2$;
- $[y_i, z_{j0}] = \binom{i+j+1}{i} y_{i+j}$ for $0 \leq i, j \leq 2^{m_1}-2$;
- $[y_i, z_{jk}] = \binom{i+j+1}{i} z_{i+j, k-1}$ for $0 \leq i, j \leq 2^{m_1}-2, 0 < k \leq 2^{m_2}-2$.

These calculations imply that $\langle x_j, y_i z_{lk} \mid 0 \leq i, l \leq 2^{m_1}-2, 0 \leq j, k \leq 2^{m_2}-2, (l, k) \neq (2^{m_1}-2, 2^{m_2}-2) \rangle \leq S'$. Thus S' has codimension at least 1 in S . A detailed inspection of the structure constants of S further shows that $w := z_{2^{m_1}-2, 2^{m_2}-2} \notin S'$ and thus S' has codimension exactly 1 in S .

(a) We consider the case $1 \notin m = (m_1, m_2)$. Our aim is to show that S' is simple. The proof is very similar to that for the Witt algebra, except that a_{2^l-1} there is replaced by $w = z_{2^{m_1}-2, 2^{m_2}-2}$ here. Let I be a non-zero ideal of S' . Using the above commutators, it is sufficient to show that $x_0 \in I$ to obtain that $I = S'$. Let $v = \sum_{i=0}^{2^{m_2}-2} \lambda_i x_i + \sum_{j=0}^{2^{m_1}-2} \mu_j y_j + \sum_{i=0}^{2^{m_1}-2} \sum_{j=0}^{2^{m_2}-2} \nu_{ij} z_{ij}$ be a non-zero element in I .

Suppose first that all coefficients ν_{ij} are zero. If $\lambda_i \neq 0$ for some i , then let k be the

greatest such that $\lambda_k \neq 0$. Then

$$(\text{ad} y_0)^k(v) = (\text{ad} y_0)^k \left(\sum_{i=0}^k \lambda_i x_i + \sum_{j=0}^{2^{m_1}-2} \mu_j y_j \right) = \sum_{i=0}^k \lambda_i x_{i-k} = \lambda_k x_0,$$

hence $x_0 = \lambda_k^{-1} \text{ad} y_0(v) \in I$. If $\lambda_i = 0$ for all i , then let k be the greatest such that $y_k \neq 0$. Then

$$\text{ad} x_1(\text{ad} x_0)^k(v) = \text{ad} x_1(\mu_k y_0) = \mu_k x_0,$$

hence $x_0 = \mu_k^{-1} \text{ad} x_1(\text{ad} x_0)^k(v) \in I$. This proves the claim if all coefficients ν_{ij} are zero. The other case that there is a non-zero ν_{ij} can be proved by similar techniques; we leave this to the reader.

(b) Now we consider the case that $m = (1, m_2)$ with $m_2 \neq 1$. In this case S' has the basis $\{x_0, \dots, x_{2^{m_2}-2}, y_0, z_{00}, \dots, z_{0, 2^{m_2}-3}\}$. We define $\phi : S' \rightarrow W(1, (m_2))'$ by $\phi(x_i) = 0$ and $\phi(y_0) = a_0$ and $\phi(z_{0i}) = a_{i+1}$. It is technical, but not difficult to verify that this is an epimorphism of Lie algebras whose kernel $\langle x_i \mid 0 \leq i \leq 2^{m_2} - 2 \rangle$ is an abelian ideal in S . Hence $S'/N(S) \cong W(1, (m_2))'$ is simple. \bullet

5 The Hamiltonian algebra

We assume that $n \geq 2$ is even and set $n = 2r$. For $1 \leq j \leq r$ let $\sigma(j) = 1$ and $j' = j + r$. For $r < j \leq n$ let $\sigma(j) = -1$ and $j' = j - r$. Define

$$D_H : A(n, m) \rightarrow W(n, m) : f \mapsto \sum_{j=1}^n \sigma(j) D_j(f) D_{j'}.$$

Then the kernel of D_H is $\mathbb{F}1$ and $H(n, m)$ is defined as the derived subalgebra of the image of D_H . The algebra $H(n, m)$ is generated by the images $\{D_H(X^a) \mid 0 \leq a < \tau\}$ and has the dimension $\dim(A(n, m)) - 2$.

$H(2, (1, 1))$ is solvable. The other cases are considered in the following theorem.

4 Theorem: *Let $\text{char}(\mathbb{F}) = 2$ and $H = H(n, m)$ with $n = 2r$ even and $m \in \mathbb{N}^n$.*

- (a) *If $n > 2$, then H is simple.*
- (b) *If $n = 2$, then $H \cong S(2, m)'$.*

Proof: (a) Let $h_a := D_H(X^{(a)})$. Suppose that I is a non-zero ideal of H , and let $w = \sum_{0 < a < \tau} \lambda_a h_a$ be a non-zero element of I . Let $\Lambda(w) = \{a \mid \lambda_a \neq 0\}$. Then $w = \sum_{a \in \Lambda(w)} \lambda_a h_a$. Also let $\mu_i = \max\{a_i \mid a \in \Lambda(w)\}$ for $1 \leq i \leq n$. Then at least one μ_i is non-zero, since $w \neq 0$ and so some (non-zero) vector is in $\Lambda(w)$. Let k be minimal with $\mu_k \neq 0$. Now define $c_0 = \mu_k$ and

$$c_i = \max\{a_{k+i} \mid a \in \Lambda(w), a_j = c_j \ (k \leq j < k+i)\}.$$

Then the vector $v := \sum_{i=0}^{n-k} c_i \varepsilon_i$ is in $\Lambda(w)$.

Now

$$\text{ad} h_{\varepsilon_{i'}}^j(h_a) = \sigma(i)^j h_{a-j\varepsilon_i}.$$

In particular, if $a_i < j$, then $(\text{ad} h_{\varepsilon_{i'}})^j(h_a) = 0$. Thus

$$\left(\prod_{i=1}^{n-k} (\sigma(i))^{c_i} (\text{ad} h_{\varepsilon_{(k+i)'}})^{c_i} \right)(w) = \lambda_v h_{c_0 \varepsilon_k},$$

since all $a \in \Lambda(w)$ other than v have at least one $a_{k+i} < c_i$ for some $1 \leq i \leq n-k$, hence the only term which is not taken to zero by this action is $\lambda_v h_v$.

Now

$$(\text{ad} h_{k'})^{c_0-1}(h_{c_0 \varepsilon_k}) = \sigma(k') h_{\varepsilon_k},$$

and thus $h_{\varepsilon_k} \in I$.

Next we show that all h_{ε_i} are in I . First,

$$[h_{\varepsilon_k}, h_{\varepsilon_i + \varepsilon_{k'}}] = \sigma(k) h_{\varepsilon_i} \in I,$$

thus all h_{ε_i} with $1 \leq i \leq n$ are in I provided that $h_{\varepsilon_i + \varepsilon_{k'}} < \tau$. This holds for all $i \neq k'$, and is also true for $i = k'$ provided $m_{k'} > 1$. Suppose $m_{k'} = 1$. Then, since $n > 2$, there exists $1 \leq i \leq n$ with $i \notin \{k, k'\}$. For such i it follows that $h_{\varepsilon_i} \in I$ and $h_{\varepsilon_{k'} + i'} < \tau$ and:

$$[h_{\varepsilon_i}, h_{\varepsilon_{k'} + i'}] = \sigma(i) h_{\varepsilon_{k'}} \in I.$$

Hence all h_{ε_i} , $1 \leq i \leq n$, are in I .

Finally, we show for arbitrary $0 < a < \tau$ that $h_a \in I$. First suppose that $a \neq \tau - \varepsilon_i$ for some $1 \leq i \leq n$. Then there exists $1 \leq j \leq n$ such that $0 < a + \varepsilon_j < \tau$, and

$$[h_{\varepsilon_{j'}}, h_{a + \varepsilon_j}] = \sigma(j') h_a \in I.$$

For $1 \leq i \leq n$ let $a = \tau - \varepsilon_i$. Note that $h_{\varepsilon_i + \varepsilon_i'} \in I$ since $h_{\varepsilon_i + \varepsilon_i'} \neq \tau - \varepsilon_j$ for any $1 \leq j \leq n$. Now since

$$[h_{\varepsilon_i + \varepsilon_{i'}}, h_a] = \sigma(i)(a_{i'} - a_i) h_a = \sigma(i)(p^{m_i'} - 1 - (p^{m_i} - 2)) h_a \equiv \sigma(i) h_a \pmod{p},$$

we see that $h_a \in I$.

In summary, we obtain that $h_a \in I$ for all $0 < a < \tau$, and thus we conclude that $I = H$. Hence H is simple.

(b) By construction, we observe that $D_H(X^{(a)}) = D_1(X^{(a)})D_2 - D_2(X^{(a)})D_1$. Thus it follows that $\{D_H(X^{(a)}) \mid 0 \leq a < \tau\} = \{D_{ij}(X^{(a)}) \mid 1 \leq i < j \leq 2, 0 \leq a \leq \tau\}$. The right hand side of this equation is a basis for $S(2, m)'$ and the left hand side generates H . •

6 The contact algebra

We assume that $n \geq 3$ is odd and set $n = 2r + 1$. For $f \in A(n, m)$ and $1 \leq j \leq n - 1$ let $f_j = X^{\epsilon_j} D_n(f) + \sigma(j') D_{j'}(f)$. Further, let $f_n = 2f - \sum_{j=1}^{2r} \sigma(j) X^{\epsilon_j} f_{j'}$. Then we define

$$D_K : A(n, m) \rightarrow W(n, m) : f \mapsto \sum_{j=1}^n f_j D_j.$$

Then D_K is an injective linear map. We define $K(n, m)$ as the derived subalgebra of the image of D_K . Hence $K(n, m)$ is generated by $\{D_K(X^a) \mid 0 \leq a \leq \tau\}$ and has dimension $\dim(A(n, m))$.

Based on computational evidence, we propose the following conjecture.

5 Conjecture: *Let $\text{char}(\mathbb{F}) = 2$ and $K = K(n, m)$ for $n \in \mathbb{N}$ odd and $m \in \mathbb{N}^n$. Then $K'/N(K')$ is simple and is isomorphic to $H/N(H)$, where $H = H(n - 1, (m_1, \dots, m_{n-1}))$.*

Note that $H/N(H) = H$ unless $n = 3$ and $m_1 = 1$ (without loss of generality), in which case $H/N(H) \cong W(1, (m_2))'$.

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